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Double conductivity media: a comparison between phenomenological and homogenization approaches

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Abstract—The heat transfer model for the double conductivity medium is investigated. The medium is composed of two homogeneous materials which differ considerably in their conductivity characteristics. The aim of this paper is to compare the descriptions obtained by a phenomenological and a homogenization approach. The first one, which introduces two temperature fields, is shown to be inefficient for a large range of the phenomena. The latter gives the rigorous description. The study enables us to improve the phenomenological description. It provides an approximation which is valid for the quasi-static conditions.

1. INTRODUCTION

WE CONSIDER a two-constituent medium characterized by the conductivities which differ considerably, and we investigate transient heat transfers. For the sake of simplicity the two constituents will be assumed to be homogeneous. The modelling of such a medium at the scale of the heterogeneities is totally inefficient because of the high number of heterogeneities in macroscopic samples, that renders the calculations intractable. A classical idea is to replace the medium by a continuous one which is equivalent from the macroscopic point of view. We focus here on two different approaches.

On the one hand, the modelling of such media with double-conductivity has been investigated directly at the macroscopic scale, by using a phenomenological approach [1–3]. Two temperature fields are assumed, i.e. one for each constituent. Similar approaches have also been carried out for other diffusive processes, such as diffusion [4, 5], or flow through double-porosity porous media [6–10].

On the other hand, an effective macroscopic description for transient heat transfer in periodic composites with double-conductivity was obtained [11] by an homogenization technique using asymptotic developments. The method is based on the passage from the microscopic description—the heterogeneity scale—to the macroscopic one by using the small parameter ε, which is the ratio between the two scale characteristic lengths. The method gives the exact macroscopic description, within an approximation $O(\varepsilon)$.

Although the homogenization leads to the right answer, the modelling which is obtained appears to be a little more complex than the phenomenological one. And the latter is often considered as a good approximation for weakly transient processes. It is therefore of interest to investigate the links between them. The problem was already partially addressed in ref. [12] where the two descriptions were shown to lead to similar boundary value problems from the mathematical point of view.

At first, in Section 2, we present the phenomenological approach for the modelling of heat transfers in media with double conductivity.

In Section 3 the results are recalled obtained in ref. [11] with an homogenization technique. The reasoning follows a more recent formalism [13].

Then, Section 4 is devoted to the comparison between the two descriptions. It is shown that they are equivalent only in a particular case, for a given pulsation of the excitation. The phenomenological approach is shown to be inefficient for a large range of the phenomena.

Finally, based on the application of the homogenization approach, the phenomenological point of view is improved in Section 5 and a correct approximation for the quasi-static behaviour is provided.

2. PHENOMENOLOGICAL APPROACH

When considering double-conductivity media, the phenomenological reasoning is characterized by the introduction of two temperature fields, i.e. one temperature for each constituent [1]. These temperatures are defined at a macroscopic scale, directly, and the description is assumed to be continuous.

Because of the two different temperatures, a heat flux between the two constituents appears. Using this point of view, Rubinshtein [1] and Aifantis and Beskos [3] introduce an interconstituent heat transfer to be proportional to the temperature difference between the constituents. The composite material is made of two homogeneous constituents, one of them, denoted 1, being much more conductive than the other, denoted 2. The two constituents are occupying,

NOMENCLATURE

c_i	specific mass capacity of the medium i
k	coefficient characterizing the homogenization approach
$\langle k \rangle$	effective value of k
$\langle k \rangle_{ph}$	coefficient $\langle k \rangle$ obtained by comparison with the phenomenological approach
$\hat{K}$	memory function
$\hat{K}_{ph}$	memory function obtained by comparison with the phenomenological approach
l	characteristic microscopic length
L	characteristic macroscopic length
n	partial volume of the constituent
$\vec{n}$	unit normal to Γ
$\vec{q}_i$	heat flux for the medium i
Q_i	quantity of heat in Ω_i
t	time
t_i	temperature field of the medium i
T_i	amplitude of the harmonic perturbation of t_i
$\vec{x}$	low space variable
$\vec{y}$	fast space variable
$\langle \rho c \rangle$	effective value of ρc for the homogenization approach
$\langle \rho c \rangle_{\Omega_i}$	average of ρc in the medium i for the phenomenological approach.

Greek symbols

α	transfer coefficient for the phenomenological approach
γ	dimensionless pulsation
Γ	boundary between the two media for the homogenization approach
δ_{ij}	Kronecker delta
ε	small parameter
ζ_i	dimensionless number defined in the medium i
λ_i	conductivity of the medium i for the homogenization approach
$\bar{\lambda}_i$	macroscopic thermic conductivity of medium i for the phenomenological approach
λ_{eff}	effective conductivity for the homogenization approach
ρ_i	density of the medium i
χ_i	particular solution for T
Ψ_i	term of heat transfer between the two constituents of the composite for the phenomenological approach
ω	pulsation
Ω	spatial period
Ω_i	volume occupied by medium i.

respectively, the volumes Ω_1 and Ω_2. Two temperature fields, t_1 and t_2, are defined in each point as the average of the temperatures in the corresponding constituent, respectively. Corresponding are two heat fluxes $\vec{q}_1$ and $\vec{q}_2$ per unit surface which are again defined at each point. At each point the quantity Q_i is also defined as the quantity of heat in a volume Ω_i of the medium i, $i = 1, 2$.

The heat equation in the medium i is then written

$$\frac{dQ_i}{dt} = \int_{\Omega_i} \vec{\nabla}(\bar{\lambda}_i \vec{\nabla} t_i)\, dv + \int_{\Omega_i} \Psi_i\, dv$$

where $\bar{\lambda}_i$ denotes the macroscopic thermal conductivity of the medium i, t is time and ψ_i an internal production term, the interconstituent heat transfer. It is often supposed that this latter can be considered in a linearized form. Therefore, Ψ_1 (which is defined as the heat transfer from medium 2 to medium 1) can be put in the form

$$\Psi_1 = \alpha(t_1 - t_2)$$

where α is a negative constant and denotes the transfer coefficient expressed in W m^{-3} K^{-1}. Consequently, Ψ_2 characterizes the heat transfer from medium 1 to medium 2 and is written

$$\Psi_2 = -\Psi_1 = -\alpha(t_1 - t_2).$$

In medium 2, the macroscopic thermic conductivity $\bar{\lambda}_2$ is supposed negligible with respect to $\bar{\lambda}_1$. Then, the phenomenological approach for the conduction phenomenon in a composite composed of two materials very different in their conductivities, leads to the two following coupled equations

$$\vec{\nabla}(\bar{\lambda}_1 \vec{\nabla} t_1) = \langle \rho c \rangle_{\Omega_1} \frac{\partial t_1}{\partial t} - \alpha(t_1 - t_2) \qquad (1)$$

$$0 = \langle \rho c \rangle_{\Omega_2} \frac{\partial t_2}{\partial t} + \alpha(t_1 - t_2) \qquad (2)$$

for media 1 and 2, respectively; ρ_i and c_i denote the density and the specific mass capacity of the constituent i. $\langle \rho c \rangle_{\Omega_i}$ represents the average of ρc in the medium i. Let us define the partial volume of constituent 2 by

$$n = \frac{|\Omega_2|}{|\Omega|}$$

where

$$\Omega = \Omega_1 + \Omega_2.$$

We have

$$\langle \rho c \rangle_{\Omega_1} = (1-n)\rho_1 c_1$$

$$\langle \rho c \rangle_{\Omega_2} = n\rho_2 c_2.$$

Proceeding by Fourier's analysis, we study the system response to a harmonic perturbation of the pulsation ω

$$t_1 = T_1\,e^{i\omega t}$$

$$t_2 = T_2\,e^{i\omega t}$$

where T_1 and T_2 are the complex numbers.

Then, eliminating T_2 between (1) and (2) leads to

$$\vec{\nabla}(\bar{\lambda}_1\vec{\nabla}T_1) = \frac{-\alpha\omega^2 n^2(\rho_2 c_2)^2 + i\alpha^2\omega((1-n)\rho_1 c_1 + n\rho_2 c_2) + i\omega^3 n^2(1-n)\rho_1 c_1(\rho_2 c_2)^2}{\alpha^2 + \omega^2 n^2(\rho_2 c_2)^2}\,T_1. \quad (3)$$

Equation (3) represents the phenomenological description of the transient heat transfer process at the constant pulsation. Thus, the phenomenological description of the problem is governed by the temperature of the more conductive constituent. Note that the conductivity of the second constituent has been neglected and that it seems to be difficult to give a physical interpretation of the temperature field T_2.

3. APPLICATION OF THE HOMOGENIZATION METHOD

Unlike the phenomenological approach, the homogenization method is based on the passage from the microscopic description to the macroscopic one. The main idea of this method is to define, if possible, a fictitious homogeneous medium, that will be hereafter referred to as the homogeneous medium or the equivalent macroscopic medium. It will behave as the composite medium when submitted to the same external constraints. The description of this medium must be intrinsic to the material and the phenomenon considered. In particular, it should not depend on the macroscopic boundary conditions.

An effective macroscopic description for transient heat conduction in periodic composites has been already derived by Auriault [11]. In this paper we follow the homogenization process which was presented in ref. [13]. Consider a two-constituent composite medium, with heterogeneity sizes $O(l)$ and a large volume of this material of a dimension L. Assume a good separation of scales

$$\varepsilon = \frac{l}{L} \ll 1.$$

The composite is periodic with a period $\Omega = O(l)$. The period is composed of two parts Ω_1 and Ω_2, occupied by the constituents 1 and 2, respectively (Fig. 1).

At the initial time, the medium is in thermal equilibrium and the temperature has a constant value throughout the period. Consider a perturbation of this equilibrium, with the pulsation ω, in such a way that the wavelength is large compared to the characteristic length l of the period. The temperature perturbation is given by

$$T(\vec{x})\,e^{i\omega t}$$

where T is a function of the space coordinates

$$\vec{x} = (x_1, x_2, x_3).$$

The determination of the macroscopic laws by the homogenization method is based on the use of an asymptotic expansion of T in powers of the small parameter ε and including a double scale with characteristic lengths l and L. Due to the separation of scales, the temperature T can be written as a function of two space variables

$$T(\vec{x}, \vec{y}).$$

The variable $\vec{x}$ is the macroscopic space variable and $\vec{y} = (\vec{x}/\varepsilon)$ is the microscopic one, describing the small heterogeneities.

The temperature T is looked for in the form

$$T(\vec{x},\vec{y}) = T^0(\vec{x},\vec{y}) + \varepsilon T^1(\vec{x},\vec{y}) + \varepsilon^2 T^2(\vec{x},\vec{y}) + \cdots$$

where the T^i are periodic with respect to $\vec{y}$, with a period $\Omega^* = \Omega/\varepsilon$. For the sake of simplicity, Ω^* will be denoted Ω in that which follows.

The method consists of incorporating such an expansion into the set of equations which describes the phenomenon at the local scale and in identifying the powers of ε, while keeping in mind that $\vec{x}$ and $\vec{y}$ should be considered as independent variables. The homogenization process gives a set of equations satisfied by T^0, which in fact represents the macroscopic behaviour within an approximation $O(\varepsilon)$.

Let λ_1 and λ_2 be the conductivities of the two media, with $\lambda_2 \ll \lambda_1$. The case of interest corresponds to

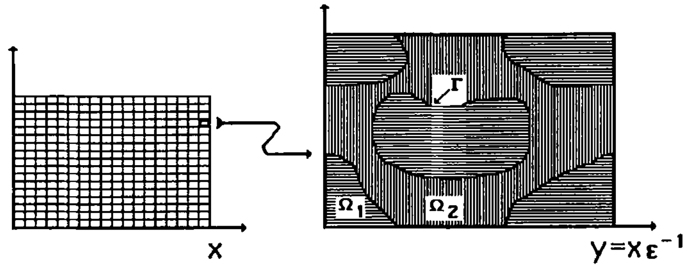

FIG. 1. Macroscopic and microscopic view of the composite.

$$\frac{\lambda_2}{\lambda_1} = O(\varepsilon^2). \quad (4)$$

The equations which govern the problem at the local scale are

$$\vec{\nabla}(\lambda_1 \vec{\nabla} T_1) = \rho_1 c_1 i\omega T_1 \quad \text{in } \Omega_1 \quad (5)$$

$$\vec{\nabla}(\lambda_2 \vec{\nabla} T_2) = \rho_2 c_2 i\omega T_2 \quad \text{in } \Omega_2 \quad (6)$$

$$[T]_\Gamma = 0 \quad (7)$$

$$[\lambda \vec{\nabla} T]_\Gamma \cdot \vec{n} = 0 \quad (8)$$

in which $\vec{n}$ denotes a unit vector, normal to Γ. We aim at discovering the equivalent macroscopic description. The non-dimensionalizing of equations (5)–(8) introduces the following dimensionless numbers

$$\zeta_1 = \left| \frac{\vec{\nabla}(\lambda_1 \vec{\nabla} T_1)}{\rho_1 c_1 \omega T_1} \right|$$

$$\zeta_{1L} = \frac{\lambda_1}{\rho_1 c_1 \omega L^2}$$

$$\zeta_{2L} = \frac{\lambda_2}{\rho_2 c_2 \omega L^2}.$$

We will assume that

$$\frac{\rho_1 c_1}{\rho_2 c_2} = O(1)$$

$$T_1 = O(T_2)$$

$$\zeta_{1L} = O(1).$$

Taking (4) into account, yields

$$\zeta_{2L} = O(\varepsilon^2).$$

We now make dimensionless the set of equations (5)–(8). For the sake of simplicity, we use the same notations for nondimensional quantities. Therefore, the dimensionless equations for the local description are as follows

$$\vec{\nabla}(\lambda_1 \vec{\nabla} T_1) = \rho_1 c_1 i\omega T_1 \quad (9)$$

$$\vec{\nabla}(\varepsilon^2 \lambda_2 \vec{\nabla} T_2) = \rho_2 c_2 i\omega T_2 \quad (10)$$

in the media Ω_1 and Ω_2, respectively, and

$$T_1 = T_2 \quad \text{on } \Gamma \quad (11)$$

$$\lambda_1 \vec{\nabla} T_1 \cdot \vec{n}_1 = \varepsilon^2 \lambda_2 \vec{\nabla} T_2 \cdot \vec{n}_1 \quad \text{on } \Gamma. \quad (12)$$

Let us perform the homogenization process (see ref. [11]). The equivalent macroscopic behaviour is expressed by

$$\nabla_x(\lambda_{\text{eff}} \nabla_x T^0) = (\rho_1 c_1 (1-n) + \rho_2 c_2 n - \rho_2 c_2 \langle k \rangle) i\omega T^0$$

$$\langle k \rangle = \frac{1}{|\Omega|} \int_{\Omega_2} k \, \mathrm{d}\Omega \quad (13)$$

where k is the solution of the following boundary value problem

$$\vec{\nabla}_y(\lambda_2 \vec{\nabla}_y k) = \rho_2 c_2 i\omega (k-1) \quad \text{in } \Omega_2$$

$$k = 0 \quad \text{on } \Gamma. \quad (14)$$

Putting

$$\langle k \rangle = \langle k \rangle_1 + i \langle k \rangle_2$$

it is shown in ref. [11] that

$$0 \leqslant \langle k \rangle_1 \leqslant n, \quad 0 \leqslant \langle k \rangle_2 \leqslant n. \quad (15)$$

Equation (13) represents the description obtained by the homogenization technique. Notice the role of the coefficient $\langle k \rangle$. We aim at comparing this result with the phenomenological one. We will therefore need more insight into this description. A way to determine the coefficient $\langle k \rangle$ follows. Considering equation (14), we first look for the eigenvalues μ_k and the associated eigenfunctions Φ_k of the Laplacian operator with the above boundary conditions

$$\vec{\nabla}_y(\lambda_2 \vec{\nabla}_y \phi_k) = -\mu_k \phi_k, \quad \text{(no summation on } k)$$

$$\phi_k = 0 \quad \text{on } \Gamma$$

$$\Phi_k \ \Omega \text{ periodic.}$$

It leads to a discrete spectrum

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots$$

and the eigenfunctions are orthogonal

$$\int_{\Omega_2} \phi_k \phi_1 \, \mathrm{d}\Omega = 0 \quad \text{if } k \neq 1.$$

We assume the eigenfunctions to be normalized

$$\int_{\Omega_2} \phi_p^2 \, \mathrm{d}\Omega = 1.$$

Now, looking for k in the form

$$k = \sum_{p=1}^{\infty} a_p \phi_p$$

we obtain

$$a_p = \int_{\Omega_2} \phi_p \, \mathrm{d}\Omega \frac{\rho_2 c_2 i\omega}{\rho_2 c_2 i\omega + \mu_p}.$$

Thus, the coefficient $\langle k \rangle$ is expressed by

$$\langle k \rangle = \frac{1}{|\Omega|} \sum_{p=1}^{\infty} \left[\int_{\Omega_2} \phi_p \, \mathrm{d}\Omega \right]^2 \frac{\rho_2 c_2 i\omega}{\rho_2 c_2 i\omega + \mu_p}.$$

For a transient excitation, the description is obtained by taking the inverse Fourier transform of equation (13)

$$\nabla_x(\lambda_{\text{eff}} \nabla_x T^0) = (\rho_1 c_1 (1-n) + \rho_2 c_2 n) \frac{\partial T^0}{\partial t}$$

$$- \rho_2 c_2 \int_{-\infty}^{t} \hat{K}(t-\tau) \frac{\partial^2 T^0}{\partial \tau^2} \, \mathrm{d}\tau \quad (16)$$

where $\hat{K}(t)$ is the inverse Fourier transform of $\langle k \rangle / i\omega$

$$\hat{K}(t) = \mathscr{F}^{-1}\left(\frac{\langle k \rangle}{i\omega}\right)$$
$$= \frac{1}{|\Omega|}\sum_{p=1}^{\infty}\left[\int_{\Omega_2}\phi_p \, d\Omega\right]^2 \exp\left(-\frac{\mu_p}{\rho_2 c_2}t\right). \quad (17)$$

Note that in the general case, $\hat{K}(t)$ is a sum of exponential terms. It represents a memory function which gives the behaviour at time t depending on the history of the second time derivative of the temperature. An equivalent formulation is obtained by integrating by parts the integral in equation (16) and noticing that all the time derivatives of $\hat{K}(t)$ are vanishing when t goes to infinity

$$\nabla_x(\lambda_{\text{eff}}\nabla_x T^0) = (\rho_1 c_1(1-n)+\rho_2 c_2 n)\frac{\partial T^0}{\partial t}$$
$$-d_2\frac{\partial^2 T^0}{\partial t^2} - d_3\frac{\partial^3 T^0}{\partial t^3} - \cdots - d_n\frac{\partial^n T^0}{\partial t^n} - \cdots. \quad (18)$$

The memory of the past is replaced by the knowledge of all the time derivatives at the present time.

4. COMPARISON BETWEEN THE TWO DESCRIPTIONS

The comparison between the phenomenological and the homogenization results is conducted by following two different ways. Firstly, we consider harmonic excitations and we show that, under certain conditions, both are equivalent for a given pulsation. Secondly, we investigate transient heat transfers. The phenomenological approach then appears as a more or less rough approximation of the quasi-static behaviour. Finally, a bilaminated composite is studied as an example. Although the significances of the temperatures which are introduced by the two approaches to describe constituent 2 do not coincide, the comparison appears to be possible because $\bar{\lambda}_1$ and T_1, which were introduced by the phenomenological approach, can be identified with λ_{eff} and T^0, respectively.

4.1. *Harmonic excitations*

Following the above remark, equations (3) and (13) will coincide when their right-hand members are equal

$$\frac{-\alpha\omega^2 n^2(\rho_2 c_2)^2 + i\alpha^2\omega((1-n)\rho_1 c_1 + n\rho_2 c_2) + i\omega^3 n^2(1-n)\rho_1 c_1(\rho_2 c_2)^2}{\alpha^2+\omega^2 n^2(\rho_2 c_2)^2}$$
$$= ((1-n)\rho_1 c_1 + n\rho_2 c_2 - \rho_2 c_2\langle k \rangle)i\omega.$$

This enables us to introduce a $\langle k \rangle$, which is denoted $\langle k \rangle_{\text{ph}}$ for the phenomenological approach. Let us put

$$\langle k \rangle_{\text{ph}} = \langle k \rangle_{1\text{ph}} + i\langle k \rangle_{2\text{ph}}.$$

Identifying the real and imaginary parts gives the two following equations

$$\langle k \rangle_{1\text{ph}} = \frac{\omega^2 n^3(\rho_2 c_2)^2}{\alpha^2+\omega^2 n^2(\rho_2 c_2)^2} \quad (19)$$

$$\langle k \rangle_{2\text{ph}} = -\frac{\alpha\omega n^2 \rho_2 c_2}{\alpha^2+\omega^2 n^2(\rho_2 c_2)^2}. \quad (20)$$

Let us introduce the dimensionless pulsation γ

$$\gamma = -\frac{n\rho_2 c_2}{\alpha}\omega.$$

Then, equations (19) and (20) become

$$\frac{\langle k \rangle_{1\text{ph}}}{n}(\gamma) = \frac{\gamma^2}{1+\gamma^2}$$

$$\frac{\langle k \rangle_{2\text{ph}}}{n}(\gamma) = \frac{\gamma}{1+\gamma^2}.$$

So, $\langle k \rangle_{1\text{ph}}/n$ and $\langle k \rangle_{2\text{ph}}/n$ can be written as a function depending on the dimensionless pulsation γ only. They are shown in Fig. 2. Note that when γ goes to zero

$$\frac{\langle k \rangle_{1\text{ph}}}{n} = O(\gamma^2)$$

and

$$\frac{\langle k \rangle_{2\text{ph}}}{n} = O(\gamma).$$

The general curve behaviours (Fig. 2) and the above results are quite similar to those obtained in ref. [11] where it is shown that $\langle k \rangle_1$ and $\langle k \rangle_2$ are, respectively, even and odd, and that when $\omega \to 0$

$$\langle k \rangle_1 = O(\omega^2), \quad \frac{\langle k \rangle_2}{\omega} = O(1).$$

The behaviours of $\langle k \rangle_{1\text{ph}}$ and $\langle k \rangle_{2\text{ph}}$ for a large pulsation are also the same

$$\frac{\langle k \rangle_{1\text{ph}}}{n} \to 1, \quad \frac{\langle k \rangle_{2\text{ph}}}{n} \to 0,$$

when

$$\omega \to \infty.$$

Let us now compare more precisely the two approaches. To ensure that the two descriptions are equivalent, equations (19) and (20) must be sim-

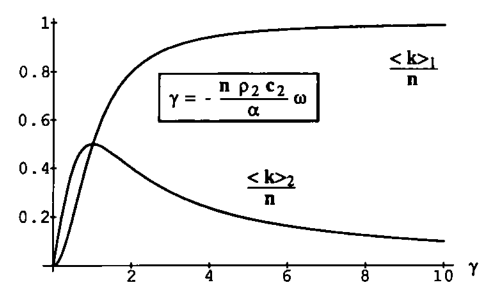

FIG. 2. Memory function: $\langle k \rangle_{1\text{ph}}/n$ and $\langle k \rangle_{2\text{ph}}/n$ with respect to the dimensionless pulsation.

ultaneously satisfied, i.e. they must lead to the same value of α.

By equating $\langle k\rangle_{ph}$ to $\langle k\rangle$, from equation (20) we obtain two possible expressions α_1 and α_2 for α

$$\alpha_1 = \frac{n\rho_2 c_2 \omega}{2}\left[-\frac{n}{\langle k\rangle_2}+\sqrt{\left(\left(\frac{n}{\langle k\rangle_2}\right)^2-4\right)}\right]$$

$$\alpha_2 = \frac{n\rho_2 c_2 \omega}{2}\left[-\frac{n}{\langle k\rangle_2}-\sqrt{\left(\left(\frac{n}{\langle k\rangle_2}\right)^2-4\right)}\right]$$

which is possible only if

$$\frac{\langle k\rangle_2}{n} \leqslant \frac{1}{2}.$$

From equation (19), α_3 is obtained

$$\alpha_3 = -n\rho_2 c_2 \omega \sqrt{\left(\frac{n}{\langle k\rangle_1}-1\right)}.$$

The two descriptions are equivalent if

$$\alpha_3 = \alpha_1$$

or if

$$\alpha_3 = \alpha_2.$$

This corresponds to a pulsation such that

$$\frac{\langle k\rangle_2}{n} = \frac{\langle k\rangle_1}{n}\sqrt{\left(\frac{1}{\frac{\langle k\rangle_1}{n}}-1\right)}.$$

Therefore, from the general behaviours of $\langle k\rangle_1$ and $\langle k\rangle_2$, as shown in Fig. 2, the above equalities will be verified, when it is possible, for a small value of the pulsation. This will be illustrated by the example of Section 4.3.

As can be seen from the general behaviours of the curves $\alpha_1(\omega)$, $\alpha_2(\omega)$ and $\alpha_3(\omega)$, this is only possible if

$$\alpha_3(0) \geqslant \alpha_2(0)$$

or if

$$\left[\frac{\langle k\rangle_1}{\omega^2}\right]^{1/2}_{\omega=0} \geqslant \left[\frac{\langle k\rangle_2}{\omega}\right]_{\omega=0}.$$

4.2. *Transient heat transfer*

From equations (19) and (20), the phenomenological approach introduces a $\langle k\rangle$, which will be denoted $\langle k\rangle_{ph}$, given by

$$\frac{\langle k\rangle_{ph}}{i\omega} = \frac{n^2\rho_2 c_2}{n\rho_2 c_2 i\omega-\alpha}.$$

Taking the inverse Fourier transform, we obtain

$$\hat{K}_{ph}(t) = n\,e^{\alpha t/n\rho_2 c_2}. \quad (21)$$

We compare this result to the one in equation (17), obtained from the homogenization process

$$\hat{K}(t) = \frac{1}{|\Omega|}\sum_{p=1}^{\infty}\left[\int_{\Omega_2}\phi_p\,d\Omega\right]^2 \exp\left(-\frac{\mu_p}{\rho_2 c_2}t\right). \quad (17)$$

One possible way to proceed is to approximate equation (17) by the first exponential $p = 1$, i.e. to limit ourselves to quasi-static excitations. Equating now the exponents, gives

$$\alpha = -n\mu_1.$$

But in general, the coefficients standing by the exponentials cannot be identified

$$\frac{1}{|\Omega|}\left[\int_{\Omega_2}\phi_1\,d\Omega\right]^2 \neq n.$$

Therefore, the phenomenological approach gives a bad approximation for quasi-static excitations.

4.3. *Bilaminated composite*

Let us consider the particular composite consisting of two homogeneous media, 1 and 2, occupying layers of respective thickness $(1-n)h/\varepsilon$ and nh/ε, measured with the space variable $\vec{y}$ (Fig. 3). For numerical purpose, we consider an academic example where medium 1 and medium 2 are composed of iron and cement, respectively. Their characteristics are as follows

$$\lambda_1 = 80.2\ \mathrm{W\,m^{-1}\,K^{-1}}$$
$$\rho_1 = 7870\ \mathrm{kg\,m^{-3}}$$
$$c_1 = 447\ \mathrm{J\,kg^{-1}\,K^{-1}}$$
$$\lambda_2 = 0.72\ \mathrm{W\,m^{-1}\,K^{-1}}$$
$$\rho_2 = 1860\ \mathrm{kg\,m^{-3}}$$
$$c_2 = 780\ \mathrm{J\,kg^{-1}\,K^{-1}}$$
$$h = 1\ \mathrm{m}.$$

When applied to this composite, ref. [11], the homogenization technique leads to

$$\langle k\rangle = n\left(1-\frac{\mathrm{th}(i^{1/2}\delta)}{i^{1/2}\delta}\right)$$

$$\delta = \left(\frac{\omega\rho_2 c_2}{\lambda_2}\right)^{1/2}\frac{nh}{2}$$

and the corresponding memory function is written in the form

$$\hat{K}(t) = 8n\sum_{p=0}^{\infty}\frac{\exp\left[-(2p+1)^2\pi^2\tau/4\right]}{(2p+1)^2\pi^2}$$

$$\tau = \frac{4\lambda_2 t}{\rho_2 c_2 n^2 h^2}.$$

One way to compare the two approaches is to identify the memory functions. The phenomenological coefficient α is no longer a constant and becomes a function of the time t, in contradiction with the phenomenological theory. Figure 4 shows the dimensionless heat transfer coefficient $\bar{\alpha}$ with respect to the

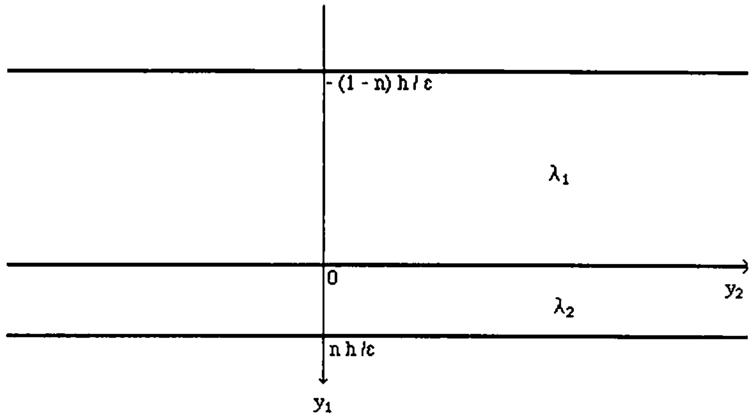

FIG. 3. Bilaminated composite.

dimensionless time τ. As τ is approaching infinity, $\bar{\alpha}$ tends to a constant. This fact will permit the derivation of the approximation presented in Section 5.

Firstly, we consider harmonic excitations. The heat transfer coefficients α_1, α_2 and α_3 are plotted in Fig. 5 against the pulsation ω for $n = 0.5$. The two approaches are equivalent for $\omega \approx 9\times10^{-6}$. On the contrary, for $n = 0.8$, there is no possible equivalence, as can be seen in Fig. 6.

Secondly, let us consider quasi-static transient heat transfers. We obtain

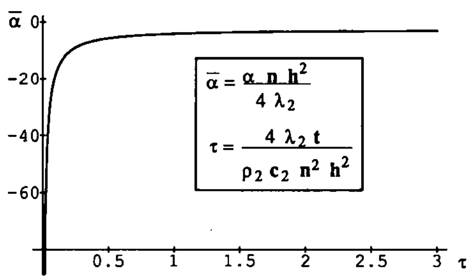

FIG. 4. Identification of the memory functions of the phenomenological and the homogenization approaches: the dimensionless heat transfer coefficient $\bar{\omega}$ against the dimensionless time τ.

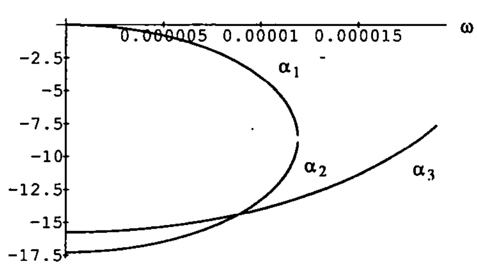

FIG. 5. Determination of the frequency for which the phenomenological and the homogenization approaches are equivalent. Bilaminated composite, $n = 0.5$.

$$\alpha = -\pi^2 \frac{\lambda_2}{nh^2}$$

and we verify that the coefficients standing by the exponentials are not equal

$$\frac{8n}{\pi^2} \neq n.$$

Although the discrepancy is small for the considered composite, it is easy to check that it can become large for different geometries.

5. AN APPROXIMATION FOR QUASI-STATIC EXCITATIONS

The phenomenological approach was shown in Section 4.2 to give a bad approximation of the modelling of quasi-static excitations. Improving the modelling implies the introduction of a second parameter to fit the homogenization result at the first order. The solution obtained by the homogenization technique, in its form (18), suggests to introduce a new term related to the time derivative of t_1 into the relation giving the heat transfer between the two constituents

$$\Psi_2 = -\Psi_1 = -\alpha\left(t_1 + \beta\frac{\partial t_1}{\partial t} - t_2\right).$$

Identifying this new model with the one obtained by

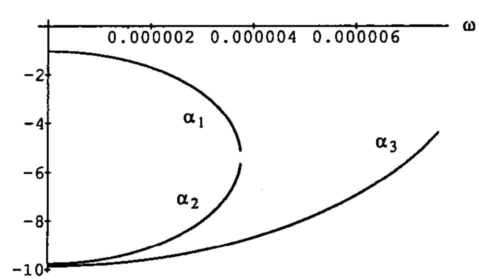

FIG. 6. As Fig. 5, $n = 0.8$.

homogenization results in new expressions $\langle k \rangle_{1ap}$ and $\langle k \rangle_{2ap}$ for $\langle k \rangle_{1ph}$ and $\langle k \rangle_{2ph}$, depending on α and β

$$\langle k \rangle_{1ph} = \frac{\omega^2 n^3 (\rho_2 c_2)^2 + \alpha\beta\omega^2 n^2 \rho_2 c_2}{\alpha^2 + \omega^2 n^2 (\rho_2 c_2)^2} = \frac{\gamma^2(\alpha\beta/\rho_2 c_2 + n)}{1+\gamma^2}$$

$$\langle k \rangle_{2ap} = -\frac{\alpha\omega n^2 \rho_2 c_2 + \alpha^2 \beta\omega n}{\alpha^2 + \omega^2 n^2 (\rho_2 c_2)^2} = \frac{\gamma(\alpha\beta/\rho_2 c_2 + n)}{1+\gamma^2}.$$

Returning to the time space, we obtain

$$\hat{K}_{ap}(t) = \left(n + \frac{\alpha\beta}{\rho_2 c_2}\right) \exp\left(\frac{\alpha}{n\rho_2 c_2} t\right).$$

It is now possible to completely identify this result with the first exponential term of the expansion of $\hat{K}(t)$. It gives

$$\alpha = -n\mu_1$$

$$\beta = -\frac{\rho_2 c_2}{n\mu_1 |\Omega|}\left[\int_\Omega \phi_1 \, d\Omega\right]^2.$$

Therefore, by taking a heat transfer term in the form

$$\Psi_2 = -\Psi_1 = -\alpha\left(t_1 + \beta\frac{\partial t_1}{\partial t} - t_2\right)$$

we obtain a correct estimation for quasi-static excitations. Note that the two coefficients α and β are directly related to the first eigenvalue and the first eigenfunction of the Laplacian operator.

The memory functions $\hat{K}_{ph}$, $\hat{K}$ and $\hat{K}_{ap}$ for the bilaminated composite presented in Section 4.3 are shown in Fig. 7 for $n = 0.8$. The value of the heat transfer coefficient α was determined from its asymptotic value, see Fig. 4.

By introducing higher derivatives of t_1 in the heat transfer term, we would have obtained a still better approximation. In the limiting case, where all derivatives are taken into consideration, the phenomenological approach leads to a similar result to the one given by the homogenization process.

With a view to comparing the three modellings, we consider a semi-infinite boundary value problem, $x \geqslant 0$, for the bilaminated composite described in Section 4.3. A temperature $T_0 \cos(\omega t)$, T_0 constant, is applied along the boundary $x = 0$. The partial volume n of constituent 2 is 0.9. The temperature is plotted against the dimensionless space variable $\bar{x}$, at $\omega t = \pi/2$, in Figs. 8–11 for different dimensionless pulsations $\bar{\omega}$

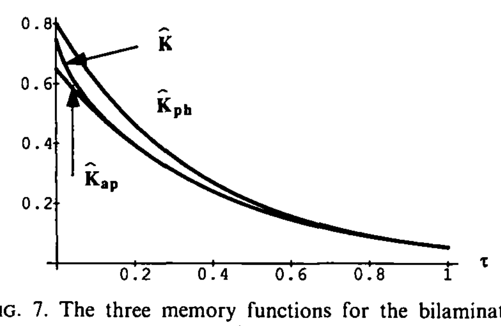

FIG. 7. The three memory functions for the bilaminated composite, $n = 0.8$.

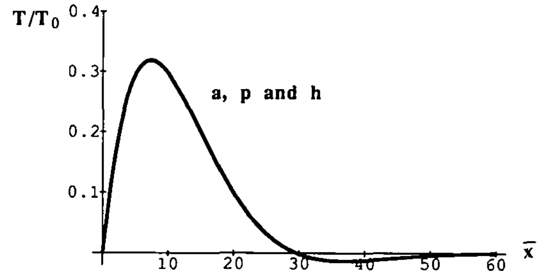

FIG. 8. Temperature profile in a semi-infinite bilaminated composite ($n = 0.9$) subjected to a cosinusoidal temperature variation at the origin, for the dimensionless pulsation $\bar{\omega} = 0.02$. h: homogenization; p: phenomenological approach; a: approximation.

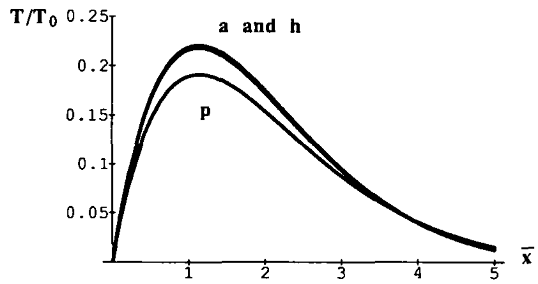

FIG. 9. As Fig. 8, $\bar{\omega} = 1$.

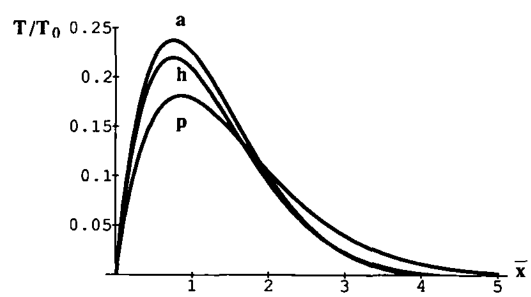

FIG. 10. As Fig. 8, $\bar{\omega} = 4$.

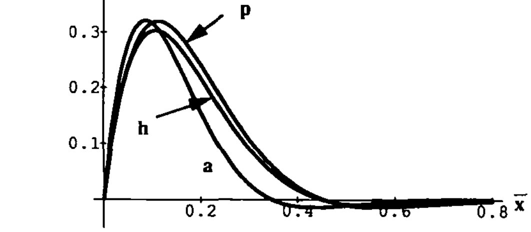

FIG. 11. As Fig. 8, $\bar{\omega} = 400$.

$$\bar{x} = \frac{\pi}{nh}\sqrt{\left(\frac{\lambda_2}{(1-n)\lambda_1}x\right)} \quad \bar{\omega} = \frac{n^2h^2\rho_2 c_2}{\pi^2\lambda_2}.$$

In Fig. 8, where $\bar{\omega}$ is very small ($\bar{\omega} = 0.02$) the discrepancy between the three modellings is negligible: the heat flux is almost permanent and the conductivity of medium 2 is ignored. As $\bar{\omega}$ is increased and reaches the value for the maximum of $\langle k \rangle_2$, $\bar{\omega} = 1$, Fig. 9, the homogenization approach and the presented approximation lead to an identical behaviour, which is different from the phenomenological one. If $\bar{\omega}$ is further increased, Fig. 10 with $\bar{\omega} = 4$, the discrepancy between all the three modellings is noticed. Nevertheless the approximation modelling still gives a better fit. Finally, as $\bar{\omega}$ is increasing to infinity (see Fig. 11 with $\bar{\omega} = 400$) the phenomenological curve tends to the homogenization one, whereas the approximation modelling curve remains apart. Notice, however, that high values of $\bar{\omega}$ correspond to nonhomogenizable situations without separation of scales and equivalent macroscopic continuous description.

6. CONCLUSION

We have investigated the heat transfer process in double-conductivity media by phenomenological and homogenization approaches. The first one, which introduces two temperature fields, leads to a description governed by the temperature of the most conductive constituent. It represents two important disadvantages: the conductivity of the less conductive constituent is neglected and is not taken into account in the macroscopic model, and the corresponding temperature field has no real physical interpretation. The second one gives the rigorous effective macroscopic description but the macroscopic model appears to be less simple. These two approaches have been compared. Firstly, by considering harmonic excitations we have shown that, under certain restrictions, both are equivalent for a given small pulsation. Secondly, investigating transient heat transfer has displayed the phenomenological description as a rough approximation for quasi-static excitations. These conclusions were confirmed by a numerical example. Finally, the phenomenological model was improved by introducing a new derivative term in the inter-constituent heat transfer. It results in an estimation for the quasi-static behaviour. The approximation would be further improved by introducing higher derivatives.

It is clear that the above analysis can be applied to other diffusive phenomena in double-diffusivity media, like double-diffusion, double-porosity or double-resistivity media subjected to Fick, Darcy or Ohm flows, respectively. The comparison between the homogenization and the phenomenological approaches would make it possible to show the range of validity of the phenomenological modellings and also to improve them.

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